TMA4170 Fourier Analysis

$$\boxed{S_N(f) = (f * D_N)}$$

Convolution: $(f * g)(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, g(x-y)\, dy$ f, g 2π-periodic

Good kernels / approximate δ's

$\{K_n(x)\}_n$ satisfying:

$$\begin{cases} (a) \ \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n = 1 \quad \forall n & \text{mass } 1 \\ (b) \ \int_{-\pi}^{\pi} |K_n| \le M \quad \forall n & \\ (c) \ \forall \delta > 0, \ \int_{\delta \le |x| \le \pi} |K_n| \xrightarrow[n\to\infty]{} 0 & \text{consentrates at } x = 0 \end{cases}$$

Lemma:

(a) f integrable, cont. at x $\Longrightarrow$ $(K_n * f)(x) \longrightarrow f(x)$

(b) $f \in C(\mathbb{S})$ $\Longrightarrow$ $(K_n * f) \longrightarrow f$ uniformly in $\mathbb{S}$

Cesàro sum

$$\sum_{k=0}^{\infty} c_k = \lim_{n\to\infty} s_n \ , \quad s_n = \sum_{k=0}^{n} c_k \ , \quad c_k \in \mathbb{C}$$

Cesàro sum: $\quad C-\sum_{k=0}^{\infty} c_k = \lim_{n\to\infty} \sigma_n \ , \qquad \sigma_n = \frac{1}{n}\sum_{k=0}^{n-1} s_k$

n^{th} Cesaro sum/mean

Lemma:

(a) $\displaystyle\sum_{k=0}^{\infty} c_k = s \implies C-\sum_{k=0}^{\infty} c_k = s$

(b) $c_k = \sigma\left(\frac{1}{k}\right)$ as $k\to\infty$ and $\displaystyle C-\sum_{k=0}^{\infty} c_k = \sigma \implies \sum_{k=0}^{\infty} c_k = \sigma$

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V vector space on $\mathbb{C}$

Inner product on V: A map $(\cdot,\cdot): V \times V \longrightarrow \mathbb{C}$ satisfying

(i) $(x, y) = \overline{(y, x)}$

(ii) $(a x_1 + b x_2, y) = a(x_1, y) + b(x_2, y)$

(iii) $x \neq 0 \Rightarrow (x, x) > 0$

Induced norm: $\|x\| = \sqrt{(x, x)}$

Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in $(V, \|\cdot\|)$ if
normed space

$\forall \varepsilon > 0 \ \exists N > 0$ such that $n, m > N \Longrightarrow \|x_n - x_m\| < \varepsilon$

$(V, \|\cdot\|)$ is complete if every Cauchy sequence has a limit in V

Hilbert space: A complete inner product space $(V, (\cdot,\cdot))$

[complete with respect to induced norm $\|\cdot\| = \sqrt{(\cdot,\cdot)}$]

Example 1: $V = \mathbb{C}^n$, $(x, y) = \sum_{i=1}^{n} x_i \overline{y_i}$, $\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$

Hilbert space

Example 2: $\ell^2(\mathbb{Z}) = \{ x = (x_i)_{i=-\infty}^{\infty} : x_i \in \mathbb{C}, \sum_{i=-\infty}^{\infty} |x_i|^2 < \infty \}$

$$(x, y) = \sum_{i=-\infty}^{\infty} x_i \overline{y_i}, \qquad \|x\| = \sqrt{(x,x)} = \sqrt{\sum_{i=-\infty}^{\infty} |x_i|^2}$$

Hilbert space